A stacky Cayley correspondence

and application to Dolbeault geometric Langlands program

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What is the Cayley correspondence?

Roughly speaking: A correspondence of interesting connected components between moduli spaces of $G^{\mathbb{R}}$-Higgs bundles on a hyperbolic Riemann surface.

More precisely, we have:

Theorem 1 (Bradlow–Collier–García-Prada–Gothen–Oliveira 2024)

There exists an injective, **open** *and* **closed** *map*

$$\Psi : \mathcal{M}_{K^{m+1}}(\widetilde{G}^{\mathbb{R}}) \times \bigoplus_i H^0(K^{l_i+1}) \longrightarrow \mathcal{M}(G^{\mathbb{R}}).$$

- We call $\operatorname{im}\Psi =$ Cayley components. Via the Non-abelian Hodge Correspondence, these consist of discrete and faithful $G^{\mathbb{R}}$ surface group representations (i.e., *higher rank Teichmüller spaces*).
- The Cayley components share the same topological type in $\pi_1(G^{\mathbb{R}})$. The Cayley correspondence provides the necessary framework to distinguish them.

Magical $\mathfrak{sl}_2$-triples

Let $\rho : \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ be an $\mathfrak{sl}_2$-subalgebra and let (f, h, e) be the generators.

The Lie algebra decomposes as an $\mathfrak{sl}_2$-module

$$\mathfrak{g} = W_0^{\oplus M_0} \oplus \bigoplus_j W_{n_j},$$

where W_{n_j} is the irreducible weight module of highest weight n_j. Let $V_{n_j} \subset W_{n_j}$ be the highest weight space for $n_j \neq 0$.

Definition 1 (BCGPGO 2024)

We say ρ is magical if the vector space involution

$$\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$$
$$X \longmapsto \begin{cases} X & \text{if } X \in W_0^{\oplus M_0} \\ (-1)^{k+1} X & \text{if } X \in \mathrm{ad}_f^k(V_{n_j}) \end{cases}$$

preserves the Lie bracket. In this case we also call θ the magical involution associated to ρ.

Hamiltonian G-spaces associated to a magical triple

- Let G be a connected simple group of adjoint type with $\mathrm{Lie}(G) = \mathfrak{g}$.
- Let $\rho : \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ be magical with generators (f, h, e), and let $C = Z_G(\rho)$.
- Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the ± 1 eigenspace decomposition of the magical involution θ, and let $H = (G^\theta)_0$ be the fixed point subgroup.

We associate two graded Hamiltonian G-spaces to ρ:

	Cayley space	**$G^{\mathbb{R}}$-Higgs space**
Space	$M_{\mathrm{Cay}} := (f + \mathfrak{g}_e) \times^{C}_{\mathfrak{c}^*} G = (f + \bigoplus V_{2m_j}) \times^C G$	$M_{G^{\mathbb{R}}} := \mathfrak{m} \times^H G$
Symplectic form	KKS[1] form on $f + \mathfrak{g}_e$	Canonical form on $T^*(H \backslash G)$
Moment map	$\mu_{\mathrm{Cay}}(X, a) = \mathrm{Ad}_{a^{-1}}(X)$	$\mu_{G^{\mathbb{R}}}(X, a) = \mathrm{Ad}_{a^{-1}}(X)$
Grading	$m_j + 1$ on V_{2m_j}	2 on $\mathfrak{m}$

Table 1: Hamiltonian G-spaces associated to magical ρ

[1] Kostant–Kirillov–Souriau

Stack of L-twisted Higgs bundles

Let Σ be a projective curve. Given a principal Γ-bundle $\mathcal{E}_0$ and a Γ-space Y, we denote the associated bundle as $Y_{\mathcal{E}_0} := Y \times^{\Gamma} \mathcal{E}_0$.

Definition 1 (Space of Sections)

The space of sections $\mathrm{Sect}(\Sigma, Y_{\mathcal{E}_0})$ is defined via the fiber product:

$$\begin{array}{ccc} \mathrm{Sect}(\Sigma, Y_{\mathcal{E}_0}) & \longrightarrow & \mathrm{Map}(\Sigma, [Y/\Gamma]) \\ \downarrow & & \downarrow \\ \{\mathcal{E}_0\} & \longrightarrow & \mathrm{Bun}_{\Gamma} \end{array}$$

Application to L-twisted Higgs Stacks:

Let $\Gamma = \mathbb{G}_m$, and let $\mathcal{E}_0 = L^{\times}$ be the frame bundle of an arbitrary line bundle $L \to \Sigma$. If $Y = \mathfrak{g}$ carries the grading action, then $Y_{\mathcal{E}_0} = \mathfrak{g}_L := \mathfrak{g} \otimes L$.

For a gauge group G, the stack of L-twisted G-Higgs bundles is defined by letting the G-bundle vary: $\mathrm{Higgs}_G^L := \mathrm{Sect}(\Sigma, [\mathfrak{g}_L/G])$.

Gaiotto's Lagrangian stack

- Let (M, ω, μ) be a graded Hamiltonian G-space where the symplectic form ω has weight 2.
- Fix a theta characteristic $K^{1/2}$ (a square root of the canonical bundle K_Σ).

Definition 2 (Gaiotto 2018, Ginzburg–Rozenblyum 2018)

The Gaiotto's Lagrangian associated to M is the stack of sections:

$$\begin{aligned} \mathrm{Lag}(M) &:= \mathrm{Sect}\left(\Sigma, [M_{K^{1/2}}/G]\right) \\ &= \left\{(\mathcal{E}, s) \;\middle|\; \mathcal{E} \in \mathrm{Bun}_G,\ s \in \mathrm{Sect}\left(\Sigma, M_{\mathcal{E}, K^{1/2}}\right)\right\} \end{aligned}$$

where $M_{\mathcal{E}, K^{1/2}}$ is the bundle associated to the $G \times \mathbb{G}_m$-space M via the G-bundle $\mathcal{E}$ and the $\mathbb{G}_m$-bundle $L = K^{1/2}$.

Global moment map

Because the local moment map $\mu : M \to \mathfrak{g}$ is homogeneous of degree 2, it induces a well-defined global map to the stack of canonical Higgs bundles:

$$\begin{aligned} \mu_M : \mathrm{Lag}(M) &\longrightarrow \mathrm{Higgs}_G \\ (\mathcal{E}, s) &\longmapsto (\mathcal{E}, \mu(s)). \end{aligned}$$

- **Functoriality:** $\mathrm{Lag}(M)$ is functorial under Hamiltonian maps.
- **Hitchin Systems:** The image of μ_M is compatible with the Hitchin map.

Openness of Cayley morphism

Let $\phi : M_{\mathrm{Cay}} \to M_{G^{\mathbb{R}}}$ be the $G \times \mathbb{G}_m$-equivariant morphism of Hamiltonian G-spaces induced by the inclusion of Slodowy slice into the (-1)-eigenspace of θ:

$$(f + \bigoplus V_{n_j}) \times G \hookrightarrow \mathfrak{m} \times G.$$

Let $\mathrm{Cay} := \mathrm{Lag}(M_{\mathrm{Cay}})$ and $\mathrm{Higgs}_{G^{\mathbb{R}}} := \mathrm{Lag}(M_{G^{\mathbb{R}}})$.

Theorem 2 (Chen–Hsiao–Y.)

Gaiotto's Lagrangian

$$\mathrm{Lag}(\phi) : \mathrm{Lag}(M_{\mathrm{Cay}}) \longrightarrow \mathrm{Lag}(M_{G^{\mathbb{R}}})$$

induces an equivalence on tangent complexes

$$d\mathrm{Lag}(\phi) : \mathbb{T}_{\mathrm{Cay}} \longrightarrow \mathrm{Lag}(\phi)^* \mathbb{T}_{\mathrm{Higgs}_{G^{\mathbb{R}}}}.$$

Closedness of Cayley morphism

Definition 3 (Valuative criterion)

Let $f : X \to Y$ be a quasi-compact morphism of algebraic stacks. We say that f is universally closed if for every DVR R with fraction field K, every diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(R) & \longrightarrow & Y \end{array}$$

admits a dotted lift étale-locally on $\mathrm{Spec}(R)$.

Theorem 3 (Chen–Hsiao–Y.)

$$\mathrm{Lag}(\phi) : \mathrm{Cay} \longrightarrow \mathrm{Higgs}_{G^{\mathbb{R}}} \text{ is universally closed.}$$

Remark on closedness of Cayley morphism

We use a different method to prove Theorem 3 than [BCGPGO 2024], which does not rely on the properness of the Hitchin map. The idea of the proof is as follows.

- It is easy to check that

$$\mathrm{Map}(\Sigma, [(f + \bigoplus V_{n_j})/C]) \longrightarrow \mathrm{Map}(\Sigma, [\mathfrak{m}/C])$$

  is a closed substack.

- Let $H < G$ be reductive and assume $\pi_0(H)$ is abelian. Then

$$\mathrm{Bun}_H(\Sigma) \longrightarrow \mathrm{Bun}_G(\Sigma)$$

  is universally closed. We use a method borrowed from the proof of a semistable reduction theorem [Balaji–Seshadri 2002, Balaji–Parameswaran 2003].

Dolbeault geometric Langlands program

Theorem 2 (Donagi–Pantev 2012)

Fourier–Mukai duality along the Hitchin fibers induces an equivalence over the regular locus:

$$\mathbf{S}_{\mathrm{Dol}} : \mathrm{QC}(\mathrm{Higgs}_G^{\diamond}/\mathcal{A}_G^{\diamond}) \xrightarrow{\sim} \mathrm{QC}(\mathrm{Higgs}_{\check{G}}^{\diamond}/\mathcal{A}_{\check{G}}^{\diamond}).$$

Physical Perspective [Kapustin–Witten 2007, Gaiotto–Witten 2009]:
In the context of 4D $\mathcal{N} = 4$ Super Yang–Mills, $\mathbf{S}_{\mathrm{Dol}}$ is interpreted as an S-duality operating on boundary conditions. We expect this equivalence to exchange mirror branes:

G-side		$\check{G}$-side
(BAA)-brane	$\longleftrightarrow$	(BBB)-brane
$\mathcal{O}_L$ (L complex Lagrangian)	$\longleftrightarrow$	Hyperholomorphic sheaf

Dual of the Cayley morphism

- Let $\rho : \mathfrak{sl}_2 \to \mathfrak{g}$ be a *magical* $\mathfrak{sl}_2$ such that the associated $G^{\mathbb{R}}$ is *strictly quasi-split*.
- Let G be the adjoint group with $\mathrm{Lie}(G) = \mathfrak{g}$, and let $\check{G}$ be its Langlands dual.
- Let A_{Dol} be the Dolbeault A-twist functor. It maps a boundary condition M to a (BAA)-brane:

$$A_{\mathrm{Dol}}(M) := (\mu_M)_* \mathcal{O}_{\mathrm{Lag}(M)} \in \mathrm{QC}(\mathrm{Higgs}_G).$$

We view the Cayley morphism as a morphism of sheaves in $\mathrm{QC}(\mathrm{Higgs}_G)$:

$$A_{\mathrm{Dol}}(\phi) : A_{\mathrm{Dol}}(M_{G^{\mathbb{R}}}) \longrightarrow A_{\mathrm{Dol}}(M_{\mathrm{Cay}}).$$

Goal: Understand the S-dual of this morphism.

Dual of the Cayley morphism (continued)

Conjecture 1 (Work in progress with Chen, Hsiao)

Let $\check{G}_\rho \subset \check{G}$ be the Nadler/spherical dual group of $H\backslash G$.

1. **Cayley Brane:**

$$\mathbf{S}_{\mathrm{Dol}}(A_{\mathrm{Dol}}(M_{\mathrm{Cay}})) \simeq B_{\mathrm{Dol}}(\check{M}_{\mathrm{Cay}}) = \omega_{\mathrm{Higgs}_{\check{G}_\rho}}.$$

2. $\mathrm{Higgs}_{G^{\mathbb{R}}}$ **Brane:** *Let DH_{V_ρ} be the Dirac–Higgs bundle with fiber $V_\rho = \mathrm{Std}_{\mathrm{Sp}_{2n}}, \mathrm{Vect}_{\mathrm{Spin}(2n+1)}, \mathbf{26}_{\mathrm{F}_4}$ respectively.*

$$\mathbf{S}_{\mathrm{Dol}}(A_{\mathrm{Dol}}(M_{G^{\mathbb{R}}})) \simeq B_{\mathrm{Dol}}(\check{M}_{G^{\mathbb{R}}}) = \bigoplus_{k \geq 0} \bigwedge^k \mathrm{DH}_{V_\rho}[1] = \bigoplus_{k \geq 0} \left(\mathrm{Sym}^k \mathrm{DH}_{V_\rho}\right)[k].$$

3. **The Morphism:** *The $\mathbf{S}_{\mathrm{Dol}}$-dual of $A_{\mathrm{Dol}}(\phi)$ is induced by the natural projection of the symmetric algebra onto its degree-zero component:*

$$\mathbf{S}_{\mathrm{Dol}}(A_{\mathrm{Dol}}(\phi)) : \bigoplus_{k \geq 0} \left(\mathrm{Sym}^k \mathrm{DH}_{V_\rho}\right)[k] \longrightarrow \omega_{\mathrm{Higgs}_{\check{G}_\rho}}.$$

Proof sketch: Conjecture 1.1

The rest of this talk will explain why

$$\mathbf{S}_{\mathrm{Dol}}(A_{\mathrm{Dol}}(M_{\mathrm{Cay}})) \simeq B_{\mathrm{Dol}}(\check{M}_{\mathrm{Cay}}) = \omega_{\mathrm{Higgs}_{\check{G}_\rho}}$$

where $A_{\mathrm{Dol}}(M_{\mathrm{Cay}}) = \mathcal{O}_{\mathrm{Cay}} \simeq \omega_{\mathrm{Cay}}$.

Given the assumptions, we are restricted to exactly 3 cases:

Case	**Type**	G	$G^{\mathbb{R}}$	$\check{G}$	$\check{G}_\rho$
(1)	A_{2n-1}	PGL_{2n}	$\mathrm{PU}(n, n)$	SL_{2n}	Sp_{2n}
(2)	D_{n+1}	PSO_{2n+2}	$\mathrm{PSO}(n, n+2)$	Spin_{2n+2}	Spin_{2n+1}
(3)	E_6	$\mathrm{E}_6^{\mathrm{ad}}$	$\mathrm{E}_{6(2)}^{\mathrm{ad}}$	$\mathrm{E}_6^{\mathrm{sc}}$	F_4

Table 2: Dual group data for quasi-split magical ρ

Analysis of Slodowy slice

First, we analyze the Slodowy slice $\mathfrak{v} := f + \bigoplus V_{2m_j}$ as a representation of the centralizer $C = Z_G(\rho)$.

In each case, we compute the explicit decomposition of $\mathfrak{v}$ as a graded C-representation (with the grading weights indicated in parentheses):

Case	**Type**	G	C	**Graded C-Rep $\mathfrak{v}$**
(1)	A_{2n-1}	PGL_{2n}	PGL_n	$\mathrm{Ad}_C(2) \oplus \mathbf{1}(2)$
(2)	D_{n+1}	PSO_{2n+2}	PGL_2	$\mathrm{Ad}_C(n) \oplus \bigoplus_{k=1}^{n-1} \mathbf{1}(2k)$
(3)	E_6	$\mathrm{E}_6^{\mathrm{ad}}$	PGL_3	$\mathrm{Ad}_C(4) \oplus \mathbf{1}(2) \oplus \mathbf{1}(6)$

Table 3: Slodowy data as graded C-representations

Quasi-split Cayley data

Based on the Slodowy data in Table 3, we conclude

$$\mathrm{Cay} \simeq \begin{cases} (1) & \mathrm{Higgs}^{K^2}_{\mathrm{PGL}_n} \times H^0(K^2) \\ (2) & \mathrm{Higgs}^{K^n}_{\mathrm{PGL}_2} \times \bigoplus_{k=1}^{n-1} H^0(K^{2k}) \\ (3) & \mathrm{Higgs}^{K^4}_{\mathrm{PGL}_3} \times H^0(K^2) \oplus H^0(K^6) \end{cases}.$$

Furthermore the relative Hitchin base $\mathcal{A}_{\mathrm{Cay}} := \mathrm{Sect}(\Sigma, (M_{\mathrm{Cay}} /\!\!/ G)_{K^{1/2}})$ is

$$\mathcal{A}_{\mathrm{Cay}} \simeq \begin{cases} (1) & \bigoplus_{k=1}^{n} H^0(K^{2k}) \simeq \mathcal{A}_{\mathrm{Sp}_{2n}} \\ (2) & \bigoplus_{k=1}^{n} H^0(K^{2k}) \simeq \mathcal{A}_{\mathrm{Spin}_{2n+1}} \\ (3) & H^0(K^2) \oplus H^0(K^6) \oplus H^0(K^8) \oplus H^0(K^{12}) \simeq \mathcal{A}_{\mathrm{F}_4} \end{cases}.$$

Therefore $\mathcal{A}_{\mathrm{Cay}} \simeq \mathcal{A}_{\check{G}_\rho}$. Since the Hitchin bases agree, we may apply $\mathbf{S}_{\mathrm{Dol}}$ over it.

Short exact sequence of Hitchin fibers

For any reductive group G, let $a \in \mathcal{A}_G^{\diamond}$. We denote the Hitchin fiber at a over the regular locus by $\mathcal{P}_a^G$. [Donagi–Gaitsgory 2002] proved this can be identified with an abelian variety[2].

Fix $a \in \mathcal{A}_{\mathrm{Cay}}^{\diamond} \simeq \mathcal{A}_{\check{G}_\rho}^{\diamond}$. Let $\mathcal{P}_a^{\mathrm{Cay}} := \mathcal{P}_a^G \cap \mathrm{Cay}$ be the generic Hitchin fiber of:

$$\mathrm{Cay} \simeq \mathrm{Higgs}_{\mathrm{PGL}_l}^{K^m} \times \bigoplus H^0(K^{n_j}).$$

$\mathcal{P}_a^{\mathrm{Cay}}$ can also be identified with an abelian variety.

Proposition 1

There exists a short exact sequence of abelian varieties:

$$0 \longrightarrow \mathcal{P}_a^{\mathrm{Cay}} \longrightarrow \mathcal{P}_a^G \longrightarrow (\mathcal{P}_a^{\check{G}_\rho})^\vee \longrightarrow 0.$$

Given this sequence, we can deduce our claim by tracking the Fourier–Mukai duality.

[2] We pass to coarse moduli spaces / identity components to focus on the underlying abelian varieties.

Fourier–Mukai Duality of Hitchin Fibers

Suppose $f : A_1 \to A_2$ is a morphism of abelian varieties. Let $\mathbf{S}_i : \mathrm{QC}(A_i) \xrightarrow{\sim} \mathrm{QC}(\check{A}_i)$ be the Fourier–Mukai duality for $i = 1, 2$.

There exists a dual morphism $\check{f} : \check{A}_2 \to \check{A}_1$, and the FM duality satisfies the compatibility:

$$\mathbf{S}_2 \circ f_* \simeq \check{f}^! \circ \mathbf{S}_1.$$

Let $\delta_0 = \mathbf{S}_1(\omega_{A_1})$ be the skyscraper sheaf at $0 \in \check{A}_1$. Then

$$(\mathbf{S}_2 \circ f_*)(\omega_{A_1}) \simeq \check{f}^!(\delta_0) \simeq \omega_{\ker(\check{f})}.$$

Why? The pullback of δ_0 is supported on $\check{f}^{-1}(0) = \ker(\check{f})$, and by Verdier duality.

Applying this to the fiber inclusion $f : \mathcal{P}_a^{\mathrm{Cay}} \hookrightarrow \mathcal{P}_a^G$, Proposition 1 yields $\ker(\check{f}) = \mathcal{P}_a^{\check{G}_\rho}$.

Globalizing these fiber-wise dualities over the generic Hitchin base yields:

$$\mathbf{S}_{\mathrm{Dol}} : \mathcal{O}_{\mathrm{Cay}} \longleftrightarrow \omega_{\mathrm{Higgs}_{\check{G}_\rho}}$$

Proof of Proposition: B-side

We will sketch the proof of Proposition 1 by passing to the Langlands dual side (the B-side) and establishing the dual short exact sequence of abelian varieties:

$$0 \longrightarrow \mathcal{P}_a^{\check{G}_\rho} \longrightarrow \mathcal{P}_a^{\check{G}} \longrightarrow (\mathcal{P}_a^{\mathrm{Cay}})^\vee \longrightarrow 0.$$

- Let $\mathbf{G} = \check{G}$ be the Langlands dual group. Since G is adjoint, $\mathbf{G}$ is a *simply-connected* simple algebraic group.
- There exists an outer automorphism σ on $\mathbf{G}$ such that the fixed-point subgroup is exactly the spherical dual group:
$$\mathbf{H} := \mathbf{G}^\sigma = \check{G}_\rho.$$
- This outer automorphism induces a Lie algebra involution (also denoted by σ), which corresponds to a folding of Dynkin diagrams.

B-side Dynkin diagram foldings

(1) $A_{2n-1} \to C_n$

(2) $D_{n+1} \to B_n$

(3) $E_6 \to F_4$

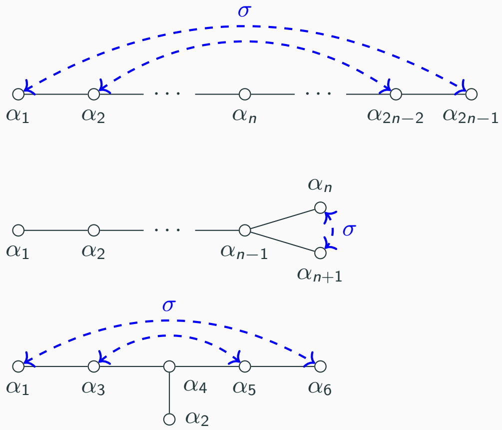


Figure 1: B-side Dynkin diagram foldings of non-adjacent roots

σ-invariant Hitchin base

Using the diagram folding, we look at the invariant geometry of the Weyl group quotients:

Lemma 1 (Slodowy 1980, Beck–Donagi–Wendland 2022)

The diagram automorphism σ induces a canonical isomorphism on the global invariant quotients:

$$\mathfrak{t}_{\mathbf{H}} /\!\!/ W_{\mathbf{H}} \xrightarrow{\sim} (\mathfrak{t}_{\mathbf{G}} /\!\!/ W_{\mathbf{G}})^{\sigma}.$$

Passing to the global stacks over Σ, this identity implies:

$$\mathcal{A}_{\mathbf{H}} \simeq \mathcal{A}_{\mathbf{G}}^{\sigma}.$$

Combining this with our earlier calculation of the Cayley base, we get

$$\mathcal{A}_{\mathrm{Cay}} = \mathcal{A}_{\check{G}_{\rho}} = \mathcal{A}_{\mathbf{H}} \simeq \mathcal{A}_{\mathbf{G}}^{\sigma} = \mathcal{A}_{\check{G}}^{\sigma}.$$

This identifies the relative Hitchin base for Cay with the σ-invariant locus of the dual total Hitchin base.

Generic Hitchin fiber as cameral data

Let $a \in (\mathcal{A}^{\diamond}_{\mathbf{G}})^{\sigma}$ be a generic σ-invariant section. Let $S^{\mathbf{G}}_a$ denote the cameral curve parametrized by a, defined by the Cartesian pullback diagram of schemes:

$$\begin{array}{ccc} S^{\mathbf{G}}_a & \longrightarrow & \mathfrak{t}_{\mathbf{G}} \otimes K \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow[a]{} & \mathfrak{t}_{\mathbf{G}} \otimes K /\!\!/ W_{\mathbf{G}} \end{array}$$

Let $\check{\Lambda}_{\mathbf{G}}$ be the cocharacter (coroot) lattice of the simply-connected group $\mathbf{G}$.

By [Donagi–Gaitsgory 2002], the generic Hitchin fiber $\mathcal{P}^{\mathbf{G}}_a$ is a *torsor* over the generalized Prym variety $(\check{\Lambda}_{\mathbf{G}} \otimes_{\mathbb{Z}} \mathrm{Jac}(S^{\mathbf{G}}_a))^{W_{\mathbf{G}}}$.

Choosing a Hitchin section as the origin trivializes the torsor into an abelian variety:

$$\mathcal{P}^{\mathbf{G}}_a \xrightarrow{\sim} \left(\check{\Lambda}_{\mathbf{G}} \otimes_{\mathbb{Z}} \mathrm{Jac}(S^{\mathbf{G}}_a)\right)^{W_{\mathbf{G}}}.$$

σ-invariant Cameral data

Lemma 2 (Beck–Donagi–Wendland 2022)

$$\mathrm{Jac}(S_a^{\mathbf{G}}) \simeq \mathsf{Ind}_{W_{\mathbf{H}}}^{W_{\mathbf{G}}}(\mathrm{Jac}(S_a^{\mathbf{H}})).$$

The proof idea is as follows. The cameral curves are ramified $W_{\mathbf{H}}$ and $W_{\mathbf{G}}$ Galois covers:

$$\begin{array}{ccc} S_a^{\mathbf{H}} & \hookrightarrow & S_a^{\mathbf{G}} \\ & \searrow \quad \swarrow & \\ & \Sigma & \end{array} .$$

Since a is σ*-invariant* and $(\mathfrak{t}_{\mathbf{G}} /\!\!/ W_{\mathbf{G}})^{\sigma} \simeq \mathfrak{t}_{\mathbf{H}} /\!\!/ W_{\mathbf{H}}$ (Lemma 1), they share the same discriminant locus over Σ. Hence the map $S_a^{\mathbf{H}} \hookrightarrow S_a^{\mathbf{G}}$ is étale onto its image, corresponding to a reduction of structure group:

$$S_a^{\mathbf{G}} \simeq \bigsqcup_{[w] \in W_{\mathbf{H}} \backslash W_{\mathbf{G}}} C_{[w]}, \text{ where } C_{[w]} \simeq S_a^{\mathbf{H}}.$$

Then applying Jacobian gives the desired claim.

Decomposition of coroot lattice

Let $\check{\Lambda}_{\mathrm{res}}$ be the coroot lattice of the restricted root system associated to $\mathbf{H}\backslash\mathbf{G}$.

Lemma 3

$$\mathrm{Res}_{W_{\mathbf{H}}}^{W_{\mathbf{G}}} \check{\Lambda}_{\mathbf{G}} \simeq \check{\Lambda}_{\mathbf{H}} \oplus \check{\Lambda}_{\mathrm{res}}.$$

Let's verify this for Case (1), where $\mathbf{G} = \mathrm{SL}_{2n}$ and $\mathbf{H} = \mathrm{Sp}_{2n}$. Let $\check{\alpha}_1, \ldots, \check{\alpha}_{2n-1}$ be the simple coroots of $\mathbf{G}$.

- **Folded coroots:** $\check{\beta}_i = \check{\alpha}_i + \check{\alpha}_{2n-i}$ (for $i < n$) and $\check{\beta}_n = \check{\alpha}_n$.
- **Restricted coroots:** $\check{\gamma}_i = \check{\alpha}_i - \check{\alpha}_{2n-i}$ (for $i < n$).

Therefore, we can explicitly decompose the $\mathbb{Z}$-module:

$$\check{\Lambda}_{\mathbf{G}} = \bigoplus_{i=1}^{2n-1} \mathbb{Z}\check{\alpha}_i = \left(\mathbb{Z}\check{\alpha}_n \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}(\check{\alpha}_i + \check{\alpha}_{2n-i})\right) \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}(\check{\alpha}_i - \check{\alpha}_{2n-i}) = \left(\mathbb{Z}\check{\beta}_n \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}\check{\beta}_i\right) \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}\check{\gamma}_i =$$

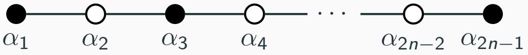
Weyl group decomposition

The proper $W_{\mathbf{H}}$-action on the sublattices is governed by the following decomposition:

Lemma 4

*$W_{\mathbf{H}} \simeq W_0 \rtimes W_{\mathrm{res}}$, where W_0 is the Weyl group of $Z_{\mathbf{H}}(\mathfrak{a})$ (generated by the **compact roots**).*

In Case (1), the real form associated to $\mathbf{H}\backslash\mathbf{G}$ is $\mathrm{SL}_n\mathbb{H}$. Its Satake diagram [Helgason 1978] is:



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- The Dynkin type of the compact part $(Z_{\mathbf{H}}(\mathfrak{a}), W_0)$ is $A_1 \times \cdots \times A_1$ (n times), hence $W_0 \simeq (\mathbb{Z}/2\mathbb{Z})^n$.
- The Dynkin type of the restricted system $(\mathrm{SL}_n\mathbb{H}, W_{\mathrm{res}})$ is A_{n-1}, thus $W_{\mathrm{res}} \simeq S_n$.

As expected, this perfectly recovers the Weyl group of $\mathbf{H} = \mathrm{Sp}_{2n}$:

$$W_{\mathbf{H}} \simeq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \simeq W_0 \rtimes W_{\mathrm{res}}.$$

Weighted invariant ring for restricted root system

Proposition 2

There exists a subspace $\mathfrak{t}_{\mathrm{res}} \subset \mathfrak{t}_{\mathbf{H}}$ equipped with a W_{res}-action such that $\mathcal{O}(\mathfrak{t}_{\mathrm{res}})^{W_{\mathrm{res}}}$ is a weighted invariant ring of type:

$$\begin{cases} (1) & A_{n-1} \text{ with weight } m = 2 \\ (2) & A_1 \text{ with weight } m = n \\ (3) & A_2 \text{ with weight } m = 4 \end{cases}.$$

We define the restricted cameral curve as $S_a^{\mathrm{res}} := S_a^{\mathbf{H}} / W_0$. It naturally fits into the following commutative diagram:

$$\begin{array}{ccc} S_a^{\mathrm{res}} & \longrightarrow & \mathfrak{t}_{\mathrm{res}} \otimes K^m \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow[a]{} & \mathfrak{t}_{\mathbf{H}} \otimes K /\!\!/ W_{\mathbf{H}} \end{array}.$$

Example of weighted invariant ring: Case (1)

Let $\mathbf{H} = \mathrm{Sp}_{2n}$, and let $x_1, \ldots, x_n$ be coordinates on $\mathfrak{t}_{\mathbf{H}}$. The Weyl group $W_{\mathbf{H}} \simeq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ acts in two steps:

- The subgroup $W_0 \simeq (\mathbb{Z}/2\mathbb{Z})^n$ acts via $x_i \mapsto \pm x_i$, yielding the basic invariants $p_i = x_i^2$.
- The quotient $W_{\mathrm{res}} \simeq S_n$ acts by permuting the vector space of generators $V = \langle p_1, \ldots, p_n \rangle$.

Let $I_{\mathrm{res}} := (V^{W_{\mathrm{res}}})$ be the ideal generated by the invariant subspace $V^{W_{\mathrm{res}}} = \langle \sum p_i \rangle$. We define the restricted subspace as:

$$\mathfrak{t}_{\mathrm{res}} := \mathsf{Spec}(\mathrm{Sym}(V)/I_{\mathrm{res}}) \simeq \mathsf{Spec}(\mathrm{Sym}(\langle p_i \mid \sum p_i = 0 \rangle)).$$

The S_n-symmetry on this hyperplane canonically forms an A_{n-1} root system.

The standard A_{n-1} invariant degrees $(2, 3, \ldots, n)$ in p_i double to $(4, 6, \ldots, 2n)$ when expressed in the underlying x_i coordinates. Thus, $\mathcal{O}(\mathfrak{t}_{\mathrm{res}})^{W_{\mathrm{res}}}$ is a **weighted invariant ring of type A_{n-1} with weight $m = 2$.**

Restricted Hitchin system

Let $\mathbf{G}^{\mathrm{res}}$ be the simply-connected group of type A_{n-1}, A_1, or A_2 corresponding to our three cases, and let $m = 2, n, 4$ be the scaling factor deduced from Proposition 2.

By construction, the restricted cover $S_a^{\mathrm{res}} \to \Sigma$ is a *ramified* W_{res} *Galois cover*, meaning it can be viewed as the cameral curve for the K^m-twisted Hitchin system $\mathrm{Higgs}_{\mathbf{G}^{\mathrm{res}}}^{K^m}$.

Note that $\mathbf{G}^{\mathrm{res}} = C^\vee$. One can explicitly verify that the twist m matches the grading of the C-representation $\mathrm{Ad}_C \subset \mathfrak{v}$ inside Cay (see Table 3).

Let $\mathcal{P}_a^{\mathrm{res}}$ be the generic Hitchin fiber of $\mathrm{Higgs}_{\mathbf{G}^{\mathrm{res}}}^{K^m}$. By *Langlands duality* for K^m-twisted PGL/SL-Higgs bundles (e.g., [Donagi–Pantev, 2012]), we obtain:

$$\mathcal{P}_a^{\mathrm{res}} \simeq (\mathcal{P}_a^{\mathrm{Cay}})^\vee.$$

Splitting of Hitchin fiber

Finally, we compute the decomposition of the global Hitchin fiber:

$$\begin{aligned}
\mathcal{P}_a^{\mathbf{G}} &= \big(\check{\Lambda}_{\mathbf{G}} \otimes \mathrm{Jac}(S_a^{\mathbf{G}})\big)^{W_{\mathbf{G}}} \\
&= \big(\check{\Lambda}_{\mathbf{G}} \otimes \mathrm{Ind}_{W_{\mathbf{H}}}^{W_{\mathbf{G}}} \mathrm{Jac}(S_a^{\mathbf{H}})\big)^{W_{\mathbf{G}}} && \text{(by Lemma 2)} \\
&= \big(\mathrm{Res}_{W_{\mathbf{H}}}^{W_{\mathbf{G}}} \check{\Lambda}_{\mathbf{G}} \otimes \mathrm{Jac}(S_a^{\mathbf{H}})\big)^{W_{\mathbf{H}}} && \text{(Projection formula + Reciprocity)} \\
&= \big((\check{\Lambda}_{\mathbf{H}} \oplus \check{\Lambda}_{\mathrm{res}}) \otimes \mathrm{Jac}(S_a^{\mathbf{H}})\big)^{W_0 \rtimes W_{\mathrm{res}}} && \text{(by Lemmas 3, 4)} \\
&= \big(\check{\Lambda}_{\mathbf{H}} \otimes \mathrm{Jac}(S_a^{\mathbf{H}})\big)^{W_{\mathbf{H}}} \times \big(\check{\Lambda}_{\mathrm{res}} \otimes \mathrm{Jac}(S_a^{\mathrm{res}})\big)^{W_{\mathrm{res}}} \\
&= \mathcal{P}_a^{\mathbf{H}} \times \mathcal{P}_a^{\mathrm{res}}.
\end{aligned}$$

Replacing $\mathbf{G} = \check{G}$, $\mathbf{H} = \check{G}_\rho$, and using the dual isomorphism $\mathcal{P}_a^{\mathrm{res}} \simeq (\mathcal{P}_a^{\mathrm{Cay}})^\vee$, we arrive at our key claim:

$$0 \longrightarrow \mathcal{P}_a^{\check{G}_\rho} \longrightarrow \mathcal{P}_a^{\check{G}} \longrightarrow (\mathcal{P}_a^{\mathrm{Cay}})^\vee \longrightarrow 0$$

is a *split short exact sequence* of abelian varieties. $\square$

Thank you!